SCARRED EIGENSTATES FOR ARITHMETIC TORAL POINT SCATTERERS

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ABSTRACT. We investigate eigenfunctions of the Laplacian perturbed by a delta potential on the standard tori $\mathbb{R}^d/2\pi\mathbb{Z}^d$ in dimensions $d = 2, 3$. Despite *quantum ergodicity* holding for the set of "new" eigenfunctions we show that there is *scarring* in the momentum representation for $d = 2, 3$, as well as in the position representation for $d = 2$ (i.e., the eigenfunctions fail to equidistribute in phase space along an infinite subsequence of new eigenvalues.) For $d = 3$, scarred eigenstates are quite rare, but for $d = 2$ scarring in the momentum representation is very common — with $N_2(x) \sim x/\sqrt{\log x}$ denoting the counting function for the new eigenvalues below x, there are $\gg N_2(x)/\log^A x$ eigenvalues corresponding to momentum scarred eigenfunctions.

1. INTRODUCTION

A basic question in Quantum Chaos is the classification of quantum limits of energy eigenstates of quantized Hamiltonians. For example, if the classical dynamics is given by the geodesic flow on a compact Riemannian manifold M, the quantized Hamiltonian is given by the positive Laplacian $-\Delta$ acting on $L^2(M)$. With $\{\psi_\lambda\}_\lambda$ denoting Laplace eigenfunctions giving an orthonormal basis for $L^2(M)$, a quantum limit is a weak* limit of $|\psi_\lambda(x)|^2$ along any subsequence of eigenvalues λ tending to infinity. More generally, given a smooth observable, i.e. a smooth function f on the unit cotangent bundle $S^*(M)$, its quantization is defined as a pseudo-differential operator $\mathrm{Op}(f)$, and one wishes to understand possible limits of the distributions

$$f \to \langle \mathrm{Op}(f)\psi_\lambda, \psi_\lambda\rangle$$

on $C^\infty(S^*(M))$, as $\lambda \to \infty$. If M has negative curvature ("strong chaos"), the celebrated Quantum Unique Ergodicity (QUE) conjecture by Rudnick and Sarnak [29] asserts that the only quantum limit is

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given by the uniform, or Liouville, measure on $S^*(M)$. Conversely, if the geodesic flow is integrable, many quantum limits may exist and the eigenfunctions are said to exhibit "scarring". E.g., if $M = \mathbb{R}^2/2\pi\mathbb{Z}^2$ is a flat torus and $a \in \mathbb{Z}$, then $\psi_a(x,y) = \cos(ax)\cos(y)$ is an eigenfunction with eigenvalue a^2+1, and clearly $|\psi_a(x,y)|^2 \xrightarrow{*} \cos^2(y)/2$ as $a \to \infty$. (For a partial classification of the set of quantum limits on $\mathbb{R}^2/2\pi\mathbb{Z}^2$, see [17].)

Now, if the flow is ergodic ("weak chaos"), Schnirelman's theorem [35, 41, 5] asserts *Quantum Ergodicity*, namely that the only quantum limit, provided we remove a zero density subset of the eigenvalues, is the uniform one. However, non-uniform quantum limits may exist along the zero density subsequence of removed eigenvalues. Some interesting questions for quantum ergodic systems are thus: are there scars? If so, how large can the exceptional set of eigenvalues be? Can eigenfunctions scar in position space, i.e., is it possible that $|\psi_\lambda(x)|^2$, along some subsequence, weakly tends to something other than $1/\operatorname{vol}(M)$? We shall address these questions for the set of "new" eigenfunctions of the Laplacian on a torus perturbed by a delta potential. The perturbation has a very small effect on the classical dynamics — only a zero measure subset of the set of trajectories is changed (hence there is no classical ergodicity), yet, as was recently shown [30, 23, 40], quantum ergodicity holds for the set of new eigenfunctions. (We note that this is quite different from point scatterers on tori of the form $\mathbb{R}^2/\Gamma$, for Γ a generic rectangular lattice. Here it was recently shown [22] that quantum ergodicity does not hold; in fact almost all new eigenfunction exhibit strong momentum scarring, cf. Section 1.2.)

1.1. **Toral point scatterers.**

The point scatterer, or the Laplacian perturbed with a delta potential (also known as a "Fermi pseudopotential"), is a popular "toy model" for studying the transition between chaos and integrability in quantum chaos. With $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$ for $d = 2$ or $d = 3$, let $\alpha \in \mathbb{R}$ denote the "strength" of a delta potential placed at some point $x_0 \in \mathbb{T}^d$; the formal operator

$$-\Delta + \alpha \cdot \delta_{x_0}$$

can then be realized using von Neumann's theory of self adjoint extensions. For $d = 2, 3$ there is a one parameter family of self adjoint extensions H_φ, parametrized by an angle $\varphi \in (-\pi, \pi]$, and the quantum dynamics we consider is generated by H_φ. For $d = 3$ we will keep φ fixed, but in order to obtain a strong spectral perturbation for $d = 2$

we will allow φ to slowly vary with the eigenvalue; in the physics literature this is known as the "strong coupling limit", cf. Section 2 for more details.

The spectrum of H_φ consists of two types of eigenvalues: "old" and "new" eigenvalues. The old ones are eigenvalues of the unperturbed Laplacian, i.e., integers that can be represented as sums of d integer squares, and the old eigenfunctions are the corresponding eigenfunctions of the unperturbed Laplacian that vanish at x_0. The set of new eigenvalues, denoted by Λ, are all of multiplicity 1, and interlace between the old eigenvalues. In fact, the new eigenvalues are solutions of the spectral equation

$$\sum_{n\in\mathcal{N}_d} r_d(n)\left(\frac{1}{n-\lambda}-\frac{n}{n^2+1}\right)=C \tag{1}$$

where

$$r_d(n):=\sum_{\xi\in\mathbb{Z}^d,\ |\xi|^2=n} 1$$

is the number of ways to represent n as a sum of d squares,

$$\mathcal{N}_d:=\{n\in\mathbb{Z}: r_d(n)>0\},$$

and

$$C=C(\varphi):=\tan(\varphi/2)\cdot\sum_n r_d(n)/(n^2+1).$$

is allowed vary with λ when $d=2$.

For $\lambda\in\Lambda$ a new eigenvalue, the corresponding eigenfunction is then given by the Green's functions $G_\lambda=(\Delta+\lambda)^{-1}\delta_{x_0}$, with L^2-expansion

$$G_\lambda(x)=-\frac{1}{4\pi^2}\sum_{\xi\in\mathbb{Z}^d}\frac{\exp(-i\xi\cdot x_0)}{|\xi|^2-\lambda}e^{i\xi\cdot x}.$$

We remark that the delta potential introduces singularities at x_0; as $x\to x_0$, we have the asymptotic (for some $a\in\mathbb{R}$)

$$G_\lambda(x)=\begin{cases} a\left(\cos(\varphi/2)\cdot\frac{\log|x-x_0|}{2\pi}+\sin(\varphi/2)\right)+o(1) & \text{for } d=2,\\ a\left(\cos(\varphi/2)\cdot\frac{-1}{4\pi|x-x_0|}+\sin(\varphi/2)\right)+o(1) & \text{for } d=3.\end{cases}$$

Note that $\varphi=\pi$ gives the unperturbed Laplacian; in what follows we will assume that $\varphi\in(-\pi,\pi)$.

We can now formulate our first result, namely that some eigenfunctions strongly localize in the momentum representation in dimension three. For $l\in\mathcal{N}_3$ let

$$\Omega(l):=\left\{\xi/|\xi|\in\mathbb{S}^2:\xi\in\mathbb{Z}^3,|\xi|^2=l\right\}$$

be the projection of the lattice points of distance $\sqrt{l}$ from the origin onto the unit sphere, and let $\delta_{\Omega(l)}$ denote the distribution defined by

$$\delta_{\Omega(l)}(f) := \frac{1}{r_3(l)} \sum_{\substack{\xi \in \mathbb{Z}^3 \\ |\xi|^2 = l}} f\left(\frac{\xi}{|\xi|}\right), \qquad \text{for } f \in C^\infty(\mathbb{S}^2)$$

(we can view it as the uniform probability measure on the points of $\Omega(l)$), and let ν denote the uniform measure on $\mathbb{S}^2$.

Theorem 1. *Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$, $x_0 \in \mathbb{T}^3$ and let Λ be the set of "new" eigenvalues of the point scatterer, that is $\Lambda = \mathrm{Spec}(\mathrm{H}_\varphi) \setminus \mathcal{N}_\mathrm{d}$. For $\lambda \in \Lambda$, let $g_\lambda \in L^2(\mathbb{T}^2)$ denote the L^2-normalized eigenfunction with eigenvalue λ. Then for any $l \in \mathcal{N}_3$ there exists an infinite subset $\Lambda_l \subset \Lambda$, and $a \in [\frac{1}{2}, 1]$ such that for any pure momentum observable $f \in C^\infty(\mathbb{S}^2)$*

$$\lim_{\lambda \in \Lambda_l} \langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle = a \cdot \delta_{\Omega(l)}(f) + (1-a) \cdot \nu(f) \tag{2}$$

That is, the pushforward of the quantum limit along this sequence to the momentum space is a convex sum of the normalized sum of delta measures on the finite set $\Omega(l)$, and the uniform measure, with at least half the mass on the singular part — there is **strong** *scarring in the momentum representation.*

In dimension 2, when φ is fixed, (1) is often referred to as the "weak coupling limit", and almost all new eigenvalues remain close to the old eigenvalues (cf. [31]). To find a model which exhibits level repulsion, Shigehara [34] and later Bogomolny and Gerland [4] considered another quantization, sometimes referred to as the "strong coupling limit". One way to arrive at this quantization is by considering energy levels in a window around a given eigenvalue: e.g., for $\eta \in (131/146, 1)$ the new eigenvalues are defined to be solutions of

$$\sum_{\substack{n \in \mathcal{N}_d \\ |n - n_+(\lambda)| < n_+(\lambda)^\eta}} r_d(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) = 0, \tag{3}$$

where $n_+(\lambda)$ is the smallest element of $\mathcal{N}_2$ that is larger than λ. It is convenient to consider both couplings simultaneously; we may do this by letting

$$F(\lambda) = \begin{cases} \text{Constant} & \text{(weak coupling)} \\ \sum\limits_{\substack{n \in \mathcal{N}_2 \\ |n - n_+(\lambda)| \geq n_+(\lambda)^\eta}} r_2(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) & \text{(strong coupling)} \end{cases} \tag{4}$$

and then rewriting the spectral equation as

$$\sum_{n\in\mathcal{N}_2} r_2(n)\left(\frac{1}{n-\lambda}-\frac{n}{n^2+1}\right)=F(\lambda) \tag{5}$$

Our next result, valid for both the weak and strong coupling limit in dimension two, is the existence of a zero density subsequence exhibiting non-uniform quantum limits in the momentum, as well as the position representation.

Theorem 2. *Let $\mathbb{T}^2=\mathbb{R}^2/2\pi\mathbb{Z}^2$, $x_0\in\mathbb{T}^2$ and let Λ be the set of new eigenvalues of the point scatterer, that is $\Lambda=\mathrm{Spec}(\mathrm{H}_\varphi)\backslash\mathcal{N}_2$. For $\lambda\in\Lambda$, let $g_\lambda\in L^2(S^*\mathbb{T}^2)$ be the L^2-normalized eigenfunction with eigenvalue λ.*

(1) There exists an infinite subset $\Lambda'_m\subset\Lambda$ such that the pushforward of the quantum limit along this sequence to momentum space has positive mass on a finite number of atoms ("strong momentum scarring").

(2) There exists an infinite subset $\Lambda'_p\subset\Lambda$ such that the pushforward of the quantum limit along this sequence to position space has a nontrivial non-zero Fourier coefficient ("position scarring").

Furthermore, we may take $\Lambda'_p=\Lambda'_m$.

We remark that for $d=2$ almost half the mass is carried on the singular part in the momentum representation — for any $\epsilon>0$ the singular part has mass at least $1/2-\epsilon$ (cf. Remark 11).

In order to quantify how common scars are we need some further notation. For $d=2,3$, let $N_d(x)$ denote the counting function ("Weyl's law") for the number of new eigenvalues $\lambda\le x$. For $d=3$, $N_3(x)\sim x$ and, as the eigenvalues that give rise to scars are essentially powers 4^l, the exceptional subset is of size $x^{o(1)}$ and thus very sparse. For $d=2$, $N_2(x)\sim x/\sqrt{\log x}=x^{1-o(1)}$, and our construction of eigenfunctions that scar both in position and momentum is a subset with counting function of size $x^{1/2-o(1)}$ — hence fairly rare. However, if we restrict ourselves to scarring only in the momentum representation, we can use some recent results by Maynard [27] to show that scarred eigenvalues are in fact quite common.

Theorem 3. *In dimension two there exists a subset $\Lambda''\subset\Lambda$ such that the pushforward of the quantum limit along Λ'' scars in momentum space, and*

$$|\{\lambda\in\Lambda'':\lambda\le x\}|\gg x/(\log x)^A$$

for some $A>1$.

1.2. **Discussion.** In [32] Šeba proposed quantum billiards on rectangles with irrational aspect ratio, perturbed with a delta potential, as a solvable singular model exhibiting wave chaos; in particular that the level spacings should be given by random matrix theory (GOE). Šeba and Życzkowski later noted [33] that the level spacings were not consistent with GOE, in particular large gaps are much more frequent (essentially having a Poisson distribution tail.) Shigehara subsequently found [34] that level repulsion is only present in the strong coupling limit. Recently Rudnick and Ueberschär proved [31], in dimension two, that the level spacing for the weak coupling limit is the same as the level spacings of the unperturbed Laplacian (after removing multiplicities). This in turn is conjectured to be Poissonian, and we note that a natural analogue of the prime k-tuple conjecture for integers that are sums of two squares can be shown to imply Poisson gaps [10]. In [31] the three dimensional case was also investigated and the mean displacement between new and old eigenvalues was shown to equal half the mean spacing.

In [30], Rudnick and Ueberschär proved a position space analogue of Quantum Ergodicity for the new eigenfunctions: there exists a full density subset of the new eigenvalues such that as $\lambda \to \infty$ along this subset, the only weak limit of $|\psi_\lambda(x)|^2$ is the uniform measure on $\mathbb{T}^2$. Further, in [23] the first author and Ueberschär proved an analogue of Quantum Ergodicity: there exists a full density subset of the new eigenvalues such that the only quantum limit along this subset is the uniform measure on the full phase space (i.e., the unit cotangent bundle $S^*(\mathbb{T}^d)$.) This result was later shown to hold also for $d = 3$ by Yesha [40]; already in [39] he showed that *all* eigenfunctions equidistribute in the position representation.

For irrational tori, Keating, Marklof and Winn proved in [18] that there exist non-uniform quantum limits (in fact, strong momentum scarring was already observed in [3]), assuming a spectral clustering condition implied by the old eigenvalues having Poisson spacings (which in turn follows from the Berry-Tabor conjecture.) Recently the first author and Ueberschär unconditionally showed [22] that for tori having diophantine aspect ratio, essentially *all* new eigenfunctions strongly scar in the momentum representation. Recently Griffin showed [13] that similar results hold for Bloch eigenmodes (i.e., non-zero quasimomentum) for periodic point scatterers in three dimensions, provided a certain Diophantine condition on the aspect ratio holds.

1.3. **Scarring and QUE for some other models.** For Quantum Ergodic systems almost all eigenfunctions equidistribute, but in general

not much is known about the (potential) subset of expectional eigenfunctions giving non-uniform quantum limits. In some cases Quantum Unique Ergodicity is known to hold; notable examples are Hecke eigenfunctions on modular surfaces [24, 36] and "quantized cat maps" [21, 19]. For these models there exist large commuting families of "Hecke symmetries" that also commute with the quantized Hamiltonian, and it is then natural to consider joint eigenfunctions of the full family of commuting operators. Other examples arise when the underlying classical dynamics is uniquely ergodic, QUE is then "automatic", e.g., see [28, 26].

On the other hand there are Quantum Ergodic systems exhibiting scarring. For example, if Hecke symmetries are not taken into account, quantized cat maps can have very large spectral degeneracies. Using this, Faure, Nonnenmacher and de-Bievre ([9]) proved that scars occur in this model. For higher dimensional analogues of cat maps, Kelmer found a scar construction not involving spectral degeneracies, but rather certain invariant rational isotropic subspaces [19, 20].

We also note that Berkolaiko, Keating, and Winn has shown [3, 2] that simultaneous momentum and position scarring can occur for quantum star graphs, e.g., for certain star graphs with a fixed (but arbitrarily large) number of bonds, there exists quantum limits supported only on two bonds.

Another way to construct scars is to use "bouncing ball quasimodes". For example, functions of the form $\psi_n(x, y) = f(x)\sin(ny)$ are approximate Laplace eigenfunctions on a stadium shaped domain (say with Dirichlet boundary conditions), and semiclassically localize on vertical periodic trajectories. Hassell showed [16] that for a generic aspect ratio stadium, there are few eigenvalues near n and hence ψ_n overlaps strongly with an eigenfunction ϕ_n with eigenvalue near n, which then also must partially localize on vertical periodic trajectories. The number of "bouncing ball eigenfunctions" having eigenvalue at most E grows (at most) as $E^{1/2+o(1)}$, to be compared with the Weyl asymptotic $c \cdot E$; hence these scarred eigenstates are fairly rare. In [1, 37, 25] the asymptotic behaviour of sets of bouncing ball eigenfunctions for some ergodic billiards were considered. Interestingly, for the stadium billiard it was argued that the number of scarred bouncing ball eigenfunctions, with eigenvalue at most E, are much more numerous, namely of order $E^{3/4}$ (again to be compared the Weyl asymptotic $c \cdot E$.) In fact, in [1] it was argued that given any $\delta \in (1/2, 1)$, there exists a Sinai type billiard whose bouncing ball eigenfunction count is of order $c_\delta \cdot E^\delta$.

1.4. **Outline of the proofs.** The proofs are based on finding new eigenvalues λ that are quite near certain old eigenvalues. After rewriting equation (5) as

$$\frac{r_d(m)}{m-\lambda} - \frac{m}{m^2+1} + H_m(\lambda) = 0, \tag{6}$$

where

$$H_m(\lambda) := \sum_{\substack{n \neq m \\ n \in \mathcal{N}_d}} r_d(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) - F(\lambda),$$

we show that for any $m \in \mathcal{N}_d$ there exists a new eigenvalue λ such that $|m-\lambda| \ll \sqrt{r_d(m)/H_m'(m)}$ (though it should be emphasized that we do not know whether $\lambda > m$ or $\lambda < m$). We then find a sequence of integers m such that both $r_d(m)$ and $\sqrt{r_d(m)/H_m'(m)}$ are bounded, and thus get a control on the distance of a new eigenvalue from these m. (To find such m we use the lower bound sieve methods when $d = 2$; for $d = 3$ we find integers m for which the representation number $r_3(m)$ is very small.) We conclude by using an explicit description for the relevant eigenfunctions to compute the limits in the theorems.

The paper is organized as follows: In Section 2 we set the necessary background for the point scatterer model, then give some number theoretic background, and in Section 3 we prove some auxiliary analytic and number theoretic results needed in the proofs of our main theorems. In Sections 4 and 5 we prove Theorems 1 and 2, and Section 6 contains the proof of Theorem 3.

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2. Background

In this section we briefly review some results and definitions about point scatterers and give a short number theoretic background.

2.1. **Point scatterers on the flat torus.**

We begin with the point scatterers, and recall the definition and properties of the quantization of observables (see [30, 23] for more details; further background can be found in [38, 40].)

2.1.1. *Basic definitions and properties.* For $d = 2, 3$ we consider the restriction of the Laplacian $-\Delta$ on

$$D_0 := C_0^\infty(\mathbb{T}^d \setminus \{x_0\})$$

The restriction is symmetric though not self-adjoint, but by von Neumann's theory of self adjoint extensions there exists a one-parameter

family of self-adjoint extensions; for $\varphi \in (-\pi, \pi]$ there exists a self-adjoint extension H_φ, where the case $\varphi = \pi$ corresponds to the unperturbed Laplacian. The spectrum of H_φ consists of two types of eigenvalues and eigenfunctions:

(1) Eigenvalues of the unperturbed Laplacian, and the corresponding eigenfunctions that vanish at x_0. The multiplicities of the new eigenvalues are reduced by 1, due to the constraint of vanishing at x_0.
(2) New eigenvalues $\lambda \in \mathbb{R}$ satisfying the equation

$$\sum_{n \in \mathcal{N}_d} r_d(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) = c_0 \tan\left(\frac{\varphi}{2}\right) \tag{7}$$

For $\lambda \in \mathbb{R}$ satisfying (7), the corresponding Green's function

$$G_\lambda(x, x_0) = (\Delta + \lambda)^{-1} \delta_{x_0} = -\frac{1}{4\pi^2} \sum_{\xi \in \mathbb{Z}^d} \frac{\exp(-i\xi \cdot x_0)}{|\xi|^2 - \lambda} e^{i\xi\cdot x}, \quad x \neq x_0 \tag{8}$$

is an eigenfunction, and

$$g_\lambda(x, x_0) := \frac{G_\lambda(x, x_0)}{\|G_\lambda\|} = \frac{\displaystyle\sum_{\xi \in \mathbb{Z}^d} \frac{\exp(-i\xi \cdot x_0)}{|\xi|^2 - \lambda} e^{i\xi\cdot x}}{\left(\displaystyle\sum_{n \in \mathcal{N}_d} \frac{r_d(n)}{|n-\lambda|^2} \right)^{1/2}} \tag{9}$$

is an L^2-normalized eigenfunction.

2.1.2. *Strong coupling.* In [30] Rudnick and Ueberschär showed that for $d = 2$ the set of "new" eigenvalues "clump" with the Laplace eigenvalues, and in fact the eigenvalue spacing distribution coincides with that of the Laplacian. In [34] Shigehara, and in [4] Bogomolny, Gerland and Schmit considered another type of quantization, with the intent of finding a model that exhibits level repulsion. This quantization is sometimes referred to as the "strong coupling" (compared to the "weak coupling" given by equation (7)). One way of arriving at this quantization is by truncating the summation in (7) outside an energy window of size $O(\lambda^\eta)$ for any fixed $\eta > 131/146$. This leads to the following spectral equation for the new eigenvalues:

$$\sum_{\substack{n \in \mathcal{N}_2 \\ |n - n_+(\lambda)| < n_+(\lambda)^\eta}} r_2(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) = c_0 \tan\left(\frac{\varphi}{2}\right). \tag{10}$$

2.1.3. *Quantization of observables.* Given a smooth observable $a(x,\xi)$ on $S^*(\mathbb{T}^d) \simeq \mathbb{T}^d \times \mathbb{S}^{d-1}$ we define the quantization of it as a pseudo-differential operator $\mathrm{Op}(a) : C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$. We refer the reader to [23] for details on the 2 dimensional case, and [40] for the 3 dimensional case. We are mainly interested in either pure momentum, or pure position observables, that is $a(x,\xi) = a(\xi) \in C^\infty(\mathbb{S}^{d-1})$, or $a(x,\xi) = a(x) \in C^\infty(\mathbb{T}^d)$ respectively; this considerably simplifies the discussion of quantizing observables. Namely, given $f(x) \in C^\infty(\mathbb{T}^d)$, the action of a pure position observable $a = a(x) \in C^\infty(\mathbb{T}^d)$ is given by

$$(\mathrm{Op}(a)f)(x) = a(x)f(x), \tag{11}$$

whereas the action of a pure momentum observable $a = a(\xi) \in C^\infty(\mathbb{S}^{d-1})$ is given by

$$(\mathrm{Op}(a)f)(x) = \sum_{v\in\mathbb{Z}^d} a\left(\frac{v}{|v|}\right)\widehat{f}(v)e^{iv\cdot x}; \tag{12}$$

in particular, for pure momentum observables we have

$$\langle \mathrm{Op}(a)f, f\rangle = \sum_{v\in\mathbb{Z}^d} a\left(\frac{v}{|v|}\right)|\widehat{f}(v)|^2. \tag{13}$$

2.2. **Number theoretic background.**

2.2.1. *Integers that are sums of* 2 *or* 3 *squares.* We begin with a short summary about integers that can be represented as sums of d squares for $d = 2$ or 3.

Sums of 2 squares: It is well known (e.g., see [8]) that $r_2(n)$ is determined by the prime factorization of n. If we write

$$n = 2^{a_0}p_1^{a_1}\dots p_r^{a_r}q_1^{b_1}\dots q_l^{b_l},$$

where the p_i's are primes all $\equiv 1 \pmod 4$, and the q_i's are primes all $\equiv 3 \pmod 4$, then n is a sum of two squares if and only if all the b_i are even, and $r_2(n) = 4d(p_1^{a_1}\dots p_r^{a_r})$, where $d(\cdot)$ is the divisor function.

Sums of 3 squares: For $d = 3$, any number n that is not of the form $n = 4^a n_1$, where $4 \nmid n_1$ and $n_1 \not\equiv 7 \pmod 8$ can be represented as a sum of 3 squares. Moreover, $r_3(4n) = r_3(n)$ for any $n \in \mathbb{Z}$, and if we let $R_3(n)$ denote the number of primitive representation of n as a sum of 3 squares (that is the number of ways to write $n = x^2 + y^2 + z^2$ with x, y, z coprime), we can relate $r_3(n)$ to class numbers of quadratic imaginary fields as follows (cf. [14, Theorem 4, p. 54]):

$$r_3(n) = \sum_{d^2|n} R_3(\frac{n}{d^2}), \quad R_3(n) = \pi^{-1}G_n\sqrt{n}L(1,\chi_n), \tag{14}$$

where

$$G_n = \begin{cases} 0 & n \equiv 0,4,7 \pmod 8 \\ 16 & n \equiv 3 \pmod 8 \\ 24 & n \equiv 1,2,5,6 \pmod 8 \end{cases}.$$

and $\chi_n(m) = (-4n/m)$ is the Kronecker symbol. By a celebrated theorem of Siegel, for any $\epsilon > 0$, $L(1, \chi_n) \gg_\epsilon n^{-\epsilon}$ and thus, for $n \not\equiv 0, 4, 7 \mod 8$,

$$r_3(n) \geq R_3(n) \gg_\epsilon n^{1/2-\epsilon}. \tag{15}$$

Further, given an integer n that is a sum of 3 squares, let

$$\Omega(n) := \left\{ \frac{(x,y,z)}{\sqrt{n}} : (x,y,z) \in \mathbb{Z}^3, x^2+y^2+z^2 = n \right\} \subset \mathbb{S}^2.$$

Fomenko-Golubeva and Duke showed (see [12, 6], or [7, Lemma 2]) that the sets $\Omega(n)$ equidistribute in $\mathbb{S}^2$ as $r_3(n) \to \infty$ (or equivalently $n_1 \to \infty$) inside $\mathcal{N}_3$. Namely, there exists $\alpha > 0$, such that for any spherical harmonic $Y(x)$, there is significant cancellation in the Weyl sum

$$W_Y(n) := \sum_{\xi \in \Omega(n)} Y(\xi),$$

in the sense that

$$W_Y(n) \ll n_1^{1/2-\alpha} \ll n^{1/2-\alpha} \tag{16}$$

where the implied constant is independent of n.

2.3. **Sieve method results.** We list below some sieve results that show the existence of various infinite sequences of integers with bounded number of prime divisors. We first recall a few definitions. A positive integer n is called r-*almost prime* if n has at most r prime divisors. We denote by P_r the set of all r-almost prime integers. A finite set of polynomials $\mathcal{F} = \{F_1(x), \ldots, F_k(x)\} \subset \mathbb{Z}[x]$ is called *admissible* if $F(x) := \prod_{i=1}^k F_i(x)$ has no fixed prime divisors, that is the equation $F(x) \equiv 0 \pmod p$ has less that p solutions for any prime p. The followoing theorem combines results from [15, 11, 27]:

Theorem 4. *Let $F_1(x), \ldots, F_k(x) \in \mathbb{Z}[x]$ ($k \geq 1$) be a finite admissible set of irreducible polynomials, and let $F(x) := F_1(x) \cdots F_k(x)$. Let G denote the degree of F. Then,*

(1) [15, Theorem 10.4] *There exists an integer $R(k,G)$, such that for any $r > R(k,G)$, as $x \to \infty$,*

$$\#\{n \in \mathbb{Z}, n \leq x : F(n) \in P_r\} \gg \frac{x}{\log^k x}$$

(2) [11, Theorem 25.4] *If $k = 1$, then one can take $R(k, G) = G + 1$, and therefore as $x \to \infty$*

$$\#\{n \le x : F(n) \in P_{G+1}\} \gg \frac{x}{\log x}$$

(3) [27, Theorem 3.4] *If k is large enough, and $F_1(x), \dots, F_k(x)$ are all **linear** with positive coefficients, then as $x \to \infty$*

$$|\{n \le x : \text{ at least two of the } F_i(n),\ 1 \le i \le k, \text{ are prime}\}| \gg \frac{x}{(\log x)^k}$$

3. Auxiliary results

Before proceeding to the proofs of the main theorems, we begin with a few auxiliary results. For the benefit of the reader we note that §3.1 is relevant for all theorems, Lemma 6 is relevant for the proof of Theorem 2, and Lemma 7, and Proposition 8 are relevant for the proof of Theorem 3.

3.1. Nearby zeros.

The following simple result will be crucial in finding integers m in the old spectrum for which there exist a nearby new eigenvalue λ.

Lemma 5. *Let I be a closed symmetric interval containing zero, and let f be C^1 function on I. Let $A > 0$ be a real number, and assume that $B := \min_{\delta \in I} f'(\delta) > 0$. If $\sqrt{A/B} \in I$ there exists $\delta_0 \in [-\sqrt{A/B}, \sqrt{A/B}]$ such that*

$$f(\delta_0) = A/\delta_0.$$

Proof. Let $I^+ = I \cap [0, \infty]$, and let $I^- = I \cap [-\infty, 0]$. For $\delta \in I^+$, we have $f(\delta) \ge f(0) + B\delta$. Similarly, for $\delta \in I^-$, $f(\delta) \le f(0) + B\delta$. Thus, since $A, B > 0$, if

$$f(0) + B\delta_1 = A/\delta_1$$

for $\delta_1 \in I^+$, there exists $\delta_0 \in [0, \delta_1]$ such that $f(\delta_0) = A/\delta_0$. Similarly, if $f(0) + B\delta_1 = A/\delta_1$ for $\delta_1 \in I^-$, there exists $\delta_0 \in [\delta_1, 0]$ such that $f(\delta_0) = A/\delta_0$.

To conclude the proof it is enough to show that

$$f(0) + B\delta = A/\delta$$

has a solution in $[-\sqrt{A/B}, \sqrt{A/B}]$, but this is clear since $B\delta^2 + f(0)\delta - A = 0$ has at least one root δ_1 for which $|\delta_1| \le \sqrt{A/B}$.

□

3.2. **Sequences of sums of two squares.**

Lemma 6. *Given $\gamma \in (0, 1/10)$ there exists an infinite set $\mathcal{M}_\gamma \subset \mathcal{N}_2$ with the following properties: $\forall m \in \mathcal{M}_\gamma$*

$$m = n^2 + 1$$

for some $n \in \mathbb{Z}^+$,

$$r_2(m) \leq 32, \tag{17}$$

and

$$r_2(m + 3) \geq 10 r_2(m)/\gamma^2. \tag{18}$$

Proof. We apply part (2) of Theorem 4 in the following setting: For $K \in \mathbb{Z}$ define

$$P(K) = \prod_{\substack{p \leq K \\ p \equiv 1 \pmod 4}} p,$$

and $r(K) \in \mathbb{Z}$ solving the following congruences:

$$\begin{aligned} &r(K)^2 + 4 \equiv 0 \pmod p \text{ if } p \equiv 1 \pmod 4, p \leq K \\ &r(K) \equiv 0 \pmod 2 \end{aligned}$$

Note that the latter equation has a solution by the Chinese remainder theorem together with -4 being a quadratic residue for any prime $p \equiv 1 \pmod 4$. We may take $0 \leq r(K) < 2P(K)$ but any fixed choice will suffice. Let $f(x) = x^2 + 1$, and let $x_K(n) = 2P(K) \cdot n + r(K)$. The polynomial $F(n) := f(x_K(n))$ satisfies all the conditions of the theorem (it is irreducible and no prime divides all coefficients), and therefore there are infinitely many n such that $F(n)$ has at most 3 prime factors, and in particular $r_2(F(n)) \leq 32$. By construction, $F(n) + 3 = x_k(n)^2 + 2^2 \equiv 0 \pmod p$ for $p \leq K$ and $p \equiv 1 \pmod 4$, hence $r_2(F(n) + 3) \geq 4 \cdot 2^{\pi(K;1,4)}$, where $\pi(K; 1, 4)$ is the number of primes occurring in the product defining $P(K)$. By choosing K appropriately, we get that

$$r_2(F(n) + 3) \geq 4d(P(K)) \geq 2^{\pi(K;1,4)+2} \geq 320/\gamma^2 \geq 10 r_2(F(n))/\gamma^2$$

□

Lemma 7. *Given $H, R \geq 2$ there exists elements $0 < a_1 < a_2 \ldots < a_H$ in $\mathcal{N}_2$ such that $a_H - a_1 < H^2$, $0 < r_2(a_1) < \ldots < r_2(a_H) \ll_H R^H$, and*

$$r_2(a_{i+1}) > R \cdot r_2(a_i)$$

holds for some $0 < i < H$.

Proof. Define
$$Q_1 = Q_1(H) := \prod_{p<2H} p^{E_p}$$
where the exponents E_p are chosen as follows: let $E_p = 1$ if $p \equiv 3 \mod 4$, otherwise let E_p be the minimal integer so that $p^{E_p} > H^2$.

Further, let $q_1 < q_2 < \ldots < q_H$ be primes congruent to $1 \mod 4$, chosen so that $q_1 > 2H$, and given integer exponents $e_1, \ldots, e_H \geq 1$, define
$$Q_2 = Q_2(e_1, e_2, \ldots, e_H) := \prod_{i \leq H} q_i^{e_i}$$
and finally let $Q := Q_1 \cdot Q_2$.

By the Chinese remainder theorem we may find $\gamma \mod Q$ such that the following holds:
$$\gamma \equiv 0 \mod p^{E_p} \quad \text{if } p \equiv 1, 2 \mod 4 \text{ and } p < 2H, \tag{19}$$
$$\gamma \equiv 1 \mod p \quad \text{if } p \equiv 3 \mod 4 \text{ and } p < 2H \tag{20}$$
and for each prime $q_i | Q_2$ so that
$$q_i^{e_i} || (\gamma^2 + i^2) \tag{21}$$

Letting $d_i = (Q_1^2, \gamma^2 + i^2)$ we define polynomials $G_i \in \mathbb{Q}[t]$ by
$$G_i(t) := ((Qt + \gamma)^2 + i^2)/(q_i^{e_i} d_i), \quad i = 1, 2, \ldots, H.$$
By definition, $d_i | \gamma^2 + i^2$, and (21) implies that $q_i^{e_i} | \gamma^2 + i^2$. Thus, since $(Q_1, Q_2) = 1$ implies that $(q_i, d_i) = 1$, we find that $q_i^{e_i} d_i | \gamma^2 + i^2$ and consequently $G_i(t) \in \mathbb{Z}[t]$ for all i.

Claim. *$\{G_i\}_{i=1}^H$ is an admissible set of polynomials (i.e., $\prod_{i=1}^H G_i(x)$ does not have any fixed prime divisors).*

To prove the claim we argue as follows: If $p > 2H$ and all G_i are nonconstant modulo p (i.e., $p \nmid Q$) there are at most $2H$ residues n (modulo p) for which $G_i(n) \equiv 0 \mod p$ for some i. Hence there exist $n \in \mathbb{Z}$ such that $\prod_{i=1}^H G_i(n) \not\equiv 0 \mod p$.

On the other hand, if $p > 2H$ and $p|Q$ then $p = q_i$ for some i, and by the definition of G_i (in particular, recall (21)), we find that $G_i(n) \not\equiv 0 \mod q_i$ for all $n \in \mathbb{Z}$. Moreover, if $j \neq i$,
$$\gamma^2 + j^2 \equiv \gamma^2 + i^2 + j^2 - i^2 \equiv j^2 - i^2 \not\equiv 0 \mod q_i$$
(as $0 < |i - j| < H$, $0 < i + j < 2H$ and $q_i > 2H$), and thus $G_j(n) \not\equiv 0 \mod q_i$ for all $n \in \mathbb{Z}$.

For $p < 2H$ we argue as follows: if $p \equiv 3 \mod 4$, (20) gives that $\gamma^2 + i^2 \not\equiv 0 \mod p$ for all $i \in \mathbb{Z}$. Otherwise, $i^2 \leq H^2 < p^{E_p}$ by our choice of E_p, and since γ was chosen so that $\gamma \equiv 0 \mod p^{E_p}$ (recall (19)), we

find that $\gamma^2 + i^2 \equiv i^2 \not\equiv 0 \mod p^{2E_p}$, as $i^2 \leq H^2$ and $p^{E_p} > H^2$. Consequently $(\gamma^2 + i^2)/d_i$ is not divisible by p. The proof of the claim is concluded.

Now, given an integer $r > 0$, let P_r denote the set of integers that can be written as a product of at most r primes, as in §2.3. Since the polynomials $\{G_i(x)\}_{i=1}^H$ form an admissible set, part (1) of Theorem 4 implies that there exists some $r > 0$ (only depending on H) such that

$$\prod_{i=1}^{H} G_i(n) \in P_r$$

for infinitely many n. Given such an n, let $m_i = G_i(n)$; then each m_i is a sum of squares that in addition has at most r prime factors. Consequently, if we set $a_i = m_i \cdot q_i^{e_i} \cdot d_i$, we find that $a_i \in \mathcal{N}_2$ for all $1 \leq i \leq H$, and that

$$e_i + 1 \leq r_2(a_i) \leq 4C \cdot (e_i + 1)$$

where $C = C(H) \geq 1$ is independent of the exponents $e_1, \ldots, e_H$. Choosing $e_1, \ldots, e_H$ appropriately we can ensure that $r_2(a_{i+1}) > R \cdot r_2(a_i)$ holds for all i, as well as that $r_2(a_H) \ll_H C^H R^H \ll_H R^H$ (a somewhat better C-dependency can be obtained but we shall not need it.)

Finally, since

$$a_i = m_i q_i^{e_i} d_i = G_i(n) q_i^{e_i} d_i = (Qn + \gamma)^2 + i^2$$

we find that $a_H - a_1 = H^2 - 1 < H^2$ and the proof of Lemma 7 is concluded.

□

The following proposition might be of independent interest — using the full power of [27], the method of the proof in fact gives the following: given $k \geq 2$ and $R > 1$ there exists $A > 0$ such that, as $x \to \infty$, there are $\gg x/(\log x)^A$ integers $n \leq x$ such that $r_2(n + h_{i+1}) \geq R r_2(n + h_i)$ holds for $i = 1, \ldots, k-1$ and $0 < h_1 < h_2 < \ldots < h_k \ll_k 1$. For simplicity we only state and prove it for $k = 2$.

Proposition 8. *There exist an integer $H \geq 1$ with the following property: for all sufficiently large R there exist an integer $h \in (0, H^2)$ such that*

$$|\{n \in \mathcal{N}_2 : n \leq x, \quad 0 < r_2(n) \ll R^H, \quad r_2(n+h) \geq R \cdot r_2(n)\}| \gg_R x/(\log x)^H$$

as $x \to \infty$.

Proof. By part (3) of Theorem 4 there exists integers i, j such that $0 < i < j \leq H$ with the property that

$$|\{n \leq x : F_i(n), F_j(n) \text{ both prime}\}| \gg x/\log^H x$$

for $\{F_1, F_2, \ldots, F_H\}$ *any* admissible set of H **linear** forms, provided H is sufficiently large. For such an H, and a given (large) R, Lemma 7 shows there exists $a_1, \ldots, a_H > 0$ such that

$$r_2(a_{i+1}) \geq R \cdot r_2(a_i) > 0$$

for $1 \leq i < H$, and $r_2(a_H) \ll R^H$. If we define

$$F_i(n) := a_i \cdot n + 1$$

for $1 \leq i \leq H$ we obtain a set of H admissible linear forms (here admissibility is trivial since $F_i(0) \not\equiv 0 \mod p$ for any prime p), hence there exists i, j with $j > i$ such that

$$|\{n \leq x : F_i(n), F_j(n) \text{ both prime}\}| \gg x/\log^H x$$

Further, given primes $p = F_i(n)$ and $p' = F_j(n)$, define $m = a_j \cdot p$ and $m' = a_i \cdot p'$. Now, since $a_i \equiv 0 \mod 4$ for all i, $F_i(n) \equiv 1 \mod 4$ for all n, hence $p, p' \equiv 1 \mod 4$ and consequently $m, m' \in \mathcal{N}_2$. Further, $m' \ll_R x$; letting $h = m - m'$ we find that

$$h = m - m' = a_j \cdot F_i(n) - a_i \cdot F_j(n) = a_j - a_i$$

and thus $0 < h < H^2$. Moreover,

$$r_2(m) = r_2(p \cdot a_j) = 2 \cdot r_2(a_j)$$

and similary $r_2(m') = 2 \cdot r_2(a_i)$. Since $r_2(a_j) \geq R \cdot r_2(a_i)$ we find that

$$r_2(m') \geq R \cdot r_2(m),$$

and that $r_2(m) = 2 \cdot r_2(a_i) \ll R^H$. Taking $n = m'$ and $h = m - m'$ we find that the number of $n \ll_R x$ with the desired property is $\gg x/(\log x)^H$, thus concluding the proof. □

4. Proof of Theorem 1

We prove Theorem 1 by calculating the Fourier coefficients of the measure (or more precisely, the coefficents in the spherical harmonics expansion.) Let $Y(x)$ be a spherical harmonic on $\mathbb{S}^2$. Then by the definition of $g_\lambda = \frac{G_\lambda}{\|G_\lambda\|}$ (cf. (9)), and the action of $\mathrm{Op}(Y)$ (cf. (13)) we

get

(22) $\langle \mathrm{Op}(Y)g_\lambda, g_\lambda\rangle =$

$$= \frac{\displaystyle\sum_{\xi\in\mathbb{Z}^3} Y\left(\frac{\xi}{|\xi|}\right)\left(\frac{1}{|\xi|^2-\lambda}\right)^2}{\displaystyle\sum_{\xi\in\mathbb{Z}^3}\left(\frac{1}{|\xi|^2-\lambda}\right)^2} = \frac{\displaystyle\sum_{n\in\mathcal{N}_3} W_Y(n)\left(\frac{1}{n-\lambda}\right)^2}{\displaystyle\sum_{n\in\mathcal{N}_3} r_3(n)\left(\frac{1}{n-\lambda}\right)^2}$$

$$= \frac{\displaystyle W_Y(m) + (m-\lambda)^2 \sum_{n\in\mathcal{N}_3\setminus\{m\}} W_Y(n)\left(\frac{1}{n-\lambda}\right)^2}{\displaystyle r_3(m) + (m-\lambda)^2 \sum_{n\in\mathcal{N}_3\setminus\{m\}} r_3(n)\left(\frac{1}{n-\lambda}\right)^2}$$

for any $m \in \mathcal{N}_3$. Now, for $m \in \mathcal{N}_3$, define

$$H_m(\lambda) := \sum_{n\in\mathcal{N}_3\setminus\{m\}} r_3(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right)$$

and rewrite the "new" eigenvalue equation (7) as

$$\frac{r_3(m)}{m-\lambda} - \frac{m}{m^2+1} + H_m(\lambda) = c_0 \tan\left(\frac{\varphi}{2}\right). \tag{23}$$

We can now apply Lemma 5. Setting $\lambda = m + \delta$, let

$$f_m(\delta) := H_m(m+\delta) - \frac{m}{m^2+1} - c_0\tan\left(\frac{\varphi}{2}\right).$$

Then

$$f_m'(\delta) = \sum_{n\in\mathcal{N}_3\setminus\{m\}} \frac{r_3(n)}{(n-m-\delta)^2} > 0 \tag{24}$$

Notice that for $|\delta| < \frac{1}{2}$ there exists an absolute constant $C > 1$ such that

$$\frac{1}{C} f_m'(0) \le f_m'(\delta) \le C f_m'(0). \tag{25}$$

Equation (23) can now be rewritten as

$$f_m(\delta) = \frac{r_3(m)}{\delta}$$

hence, *provided* that we can find m for which the bound $\sqrt{Cr_3(m)/f_m'(0)} < \frac{1}{2}$ holds, we may take $I = [-1/2, 1/2]$ in Lemma 5 and obtain an eigenvalue λ such that

$$|\lambda - m| < \sqrt{Cr_3(m)/f_m'(0)}. \tag{26}$$

To find m for which the above bound is valid, we proceed as follows. For $l \in \mathcal{N}_3$ fixed, define

$$\Omega(l) := \left\{ \frac{\xi}{\|\xi\|} : \|\xi\|^2 = l, \xi \in \mathbb{Z}^3 \right\}$$

and let $\mathcal{M}_l := \left\{4^k l : k \in \mathbb{N}\right\}$. For $m \in \mathcal{M}_l$ we then have $r_3(m) = r_3(l)$, hence $r_3(m)$ is uniformly bounded; we also note that $\Omega(m) = \Omega(l)$. Since for any integer m there exists an integer $m' \not\equiv 0, 4, 7 \pmod 8$ of bounded distance from m, (15) implies that

(27)
$$f'_m(0) = \sum_{n \in \mathcal{N}_3} \frac{r_3(n)}{|n-m|^2} \geq \frac{r_3(m')}{|m'-m|^2} \gg r_3(m') \gg (m')^{1/2-\varepsilon} \gg m^{1/2-\varepsilon}.$$

Since $r_3(m)$ is uniformly bounded for $m \in \mathcal{M}_l$, we find that

$$\sqrt{Cr_3(m)/f'_m(0)} < m^{-1/4+\epsilon}$$

for all sufficiently large $m \in \mathcal{M}_l$. By the above argument, we have thus found infinitely many m for which there exist a nearby new eigenvalue λ satisfying $|m-\lambda| < \sqrt{Cr_3(l)/f'_m(0)} < m^{-1/4+\epsilon}$. In fact, using (27) we can apply Lemma 5 again, to get that (25) holds for $C = 1+O(m^{-1/4+\epsilon})$ and $\delta = O(m^{-1/4+\epsilon})$. Let Λ_l be the sequence of these eigenvalues; for $\lambda \in \Lambda_l$ we then find, upon recalling the equality in (24), and that (25) is valid since $|m-\lambda| = O(m^{-1/4+\epsilon})$, that

$$|m-\lambda|^2 \sum_{n \in \mathcal{N}_3 \setminus \{m\}} \frac{r_3(n)}{|n-\lambda|^2} \leq \frac{(1+o(m^{-1/4}))r_3(l)}{f'_m(0)} \sum_{n \in \mathcal{N}_3 \setminus \{m\}} \frac{r_3(n)}{|n-\lambda|^2} = (1+O(m^{-1/4+\epsilon}))^2 r_3(l) \tag{28}$$

which is bounded. From now on we restrict Λ_l to a subsequence such that the limit

$$A_l := \lim_{\lambda \in \Lambda_l} |m-\lambda|^2 \sum_{n \in \mathcal{N}_3 \setminus \{m\}} \frac{r_3(n)}{|n-\lambda|^2}$$

exists, and hence by (28) is bounded by $r_3(l)$. Furthermore, for any spherical harmonic Y,

$$|m-\lambda|^2 \sum_{n \in \mathcal{N}_3 \setminus \{m\}} \frac{|W_Y(n)|}{|n-\lambda|^2} \leq Cr_3(l)/f'_m(0) \sum_{n \in \mathcal{N}_3 \setminus \{m\}} \frac{|W_Y(n)|}{|n-\lambda|^2}. \tag{29}$$

We claim that the RHS converges to 0 as $m \to \infty$. To see this, write

$$\sum_{n\in\mathcal{N}_3\setminus\{m\}} \frac{|W_Y(n)|}{|n-\lambda|^2} \le \sum_{\substack{n\in\mathcal{N}_3\setminus\{m\}\\ n\le m+m^{1/3}}} \frac{|W_Y(n)|}{|n-\lambda|^2} + \sum_{\substack{n\in\mathcal{N}_3\setminus\{m\}\\ n> m+m^{1/3}}} \frac{|W_Y(n)|}{|n-\lambda|^2}.$$

For the first sum, using that $|\lambda - n| > 1/2$ for $n \neq m$ together with the bound $W_Y(n) \ll m^{1/2-\alpha}$ (using (16)) we find that for all $n \le m + m^{1/3}$ in the summand, the first sum is $\ll m^{1/2-\alpha}$. For the second sum, the mean value theorem gives that

$$W_Y(n) \ll n^{1/2-\alpha} = (n-m)^{1/2-\alpha} + O\left(\frac{m}{(n-m)^{1/2+\alpha}}\right)$$

and thus,

$$\begin{aligned}(30)\qquad & \sum_{n>m+m^{1/3}} \frac{W_Y(n)}{|n-m|^2} \ll \\ & \sum_{n-m>m^{1/3}} \frac{1}{|n-m|^{3/2+\alpha}} + O\left(\sum_{n-m>m^{1/3}} \frac{m}{|n-m|^{5/2+\alpha}}\right) \ll \\ & \qquad m^{-1/3(1/2+\alpha)} + m^{1-1/3(3/2+\alpha)} \ll m^{1/2-\alpha}.\end{aligned}$$

Hence, since $f_m'(0) \gg m^{1/2-\varepsilon}$ (cf. (27)),

$$\frac{Cr_3(l)}{f'(0)} \sum_{n\in\mathcal{N}_3} \frac{W_Y(n)}{|n-m|^2} \ll \frac{m^{1/2-\alpha}}{m^{1/2-\varepsilon}} \ll m^{-\alpha+\varepsilon} \tag{31}$$

Thus, for any fixed spherical harmonic Y, and for every $\lambda \in \Lambda_l$,

$$\langle \mathrm{Op}(Y) g_\lambda, g_\lambda\rangle = \begin{cases} \frac{W_Y(l)}{(1+A_l+o(1))r_3(l)} + O(\lambda^{-\alpha+\varepsilon}) & \text{if } Y \text{ is non trivial,} \\ 1 & \text{if } Y \text{ is trivial.}\end{cases}.$$

Since these are the spherical harmonics coefficients of the measure $\frac{1}{1+A_l}\delta_{\Omega(l)} + \frac{A_l}{1+A_l}\nu$, the proof is concluded. (Recall that ν denotes the uniform measure.)

5. Proof of Theorem 2

We start by finding a sequence of new eigenvalues lying close to the set of old eigenvalues. To do so we will again use Lemma 5. Recall that

$$F(\lambda) = \begin{cases} \text{Constant} & \text{(weak coupling)} \\ \sum\limits_{\substack{n\in\mathcal{N}_2 \\ |n-n_+(\lambda)|\ge n_+(\lambda)^\eta}} r_2(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) & \text{(strong coupling)}\end{cases} \tag{32}$$

and in analogy with the three dimensional case we define

$$H_m(\lambda) = \sum_{n\in\mathcal{N}_2\setminus\{m\}} r_2(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) - F(\lambda) = \sum_{n\in I(\lambda)\setminus\{m\}} r_2(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) \tag{33}$$

where (for some fixed $\eta > 131/146$)

$$I(\lambda) := \begin{cases} \mathcal{N}_2 \cap [n_+(\lambda) - n_+(\lambda)^\eta, n_+(\lambda) + n_+(\lambda)^\eta] & \text{(strong coupling)} \\ \mathcal{N}_2 & \text{(weak coupling)} \end{cases}$$

Proposition 9. *Let $F(\lambda)$ be as above, and given $\gamma \in (0, 1/10)$ let $\mathcal{M}_\gamma$ be the set of integers given by Lemma 6. Then, for any $m \in \mathcal{M}_\gamma$, there exists a new eigenvalue λ such that $|\lambda - m| \leq \gamma$ and*

$$H'_m(\lambda) \leq (1 + O(\gamma)) \cdot \frac{r_2(m)}{(m-\lambda)^2} \tag{34}$$

as $\gamma \to 0$.

Proof. Given $m \in M_\gamma$ we start by finding (at least one) nearby new eigenvalue. To do so, rewrite the eigenvalue equation (i.e., (7) in the weak coupling limit, or (10) in the strong coupling limit)

$$\sum_{n\in S} r_2(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) = F(\lambda),$$

as

$$r_2(m)\left(\frac{1}{m-\lambda} - \frac{m}{m^2+1}\right) + H_m(\lambda) = 0$$

Thus, with $\lambda = m + \delta$, and defining $f(\delta) = H_m(m+\delta) - r_2(m)\frac{m}{m^2+1}$, we wish to find (small) solutions to

$$f(\delta) = \frac{r_2(m)}{\delta}$$

Now, by (32), $f'(\delta)$ is always a sum of positive terms, hence we may drop all terms but one, say the one corresponding to $k = m + 3$ (recall that $m = n^2 + 1$, hence $k = n^2 + 4$ is a sum of two squares), and find that

$$f'(\delta) \geq \frac{r_2(m+3)}{((m+3) - (m+\delta))^2} = \frac{r_2(m+3)}{(3-\delta)^2} \geq \frac{r_2(m+3)}{10}$$

for $|\delta| \leq 1/10$. By Lemma 5, there exists δ_0 such that

$$f(\delta_0) = \frac{r_2(m)}{\delta_0}$$

and

$$|\delta_0| \le \sqrt{10 r_2(m)/r_2(m+3)} \le \gamma$$

Using the above estimate on δ we next show that the lower bound on $f'(\delta)$ is essentially given by the size of $H'_m(m)$. Since $m_- \le m-1$ and $m_+ \ge m+1$, we find that

$$\frac{1}{(n-(m+\delta))^2} = \frac{1+O(\gamma)}{(n-m)^2}$$

holds for all $n \in \mathcal{N}_2 \setminus \{m\}$ and $|\delta| \le \gamma$. Thus,

$$\min_{|\delta|\le\gamma} f'(\delta) = \min_{|\delta|\le\gamma} H'_m(m+\delta) = H'_m(m)(1+O(\gamma))$$

for $|\delta| \le \gamma$. Hence we may take $A = r_2(m)$ and $B = H'_m(m+\delta_0)(1+O(\gamma))$ in Lemma 5; on squaring the estimate $\delta_0 \le \sqrt{A/B}$ we find that

$$\delta_0^2 H'_m(m+\delta_0) \le (1+O(\gamma)) r_2(m)$$

(for γ small.) In particular, $\lambda = m+\delta_0$ is a new eigenvalue, and

$$H'_m(\lambda) \le \frac{r_2(m)}{(m-\lambda)^2} \cdot (1+O(\gamma))$$

□

Remark 10. *The above argument in fact gives the following: if* $r_2(m+h) \ge R \cdot r_2(m) > 0$ *for some* $0 < h < H$*, then (again for* $|\delta| < 1/10$*),*

$$f'(\delta) \gg \frac{R \cdot r_2(m)}{H^2}$$

and thus there exists a nearby new eigenvalue $\lambda = m + \delta_0$ *with* $|\delta_0| \ll H/\sqrt{R}$*, and*

$$H'_m(\lambda) \ll \frac{r_2(m)}{(m-\lambda)^2}.$$

For $\gamma \in (0, 1/10)$ let $\mathcal{M}_\gamma$ be the set given by Lemma 6, and Λ_γ be the set of corresponding new eigenvalues given by Proposition 9. By restricting to a subsequence we may assume that for any $n \in \mathcal{M}_\gamma$, the sets

$$\Xi(n) := \left\{ \frac{\xi}{|\xi|} : |\xi|^2 = n \right\}$$

converge to a limit set $\Xi(\infty)$ of bounded cardinality.

It is now straightforward to exhibit scarring in momentum space.

5.1. **Scarring in momentum space.** In this section we prove the first part of Theorem 1. For any fixed *positive* $f \in C^\infty(\mathbb{S}^1)$ we show that there exists a positive constant $0 < c \leq 1$ such that

$$\lim_{\lambda \in \Lambda'} \langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle \geq \frac{c}{|\Xi|} \sum_{\xi \in \Xi} f(\xi) \tag{35}$$

Let

$$W_f(n) := \sum_{|\xi|^2 = n} f\left(\frac{\xi}{|\xi|}\right).$$

We start with an upper bound on the L^2 norm of G_λ: By definition of G_λ, its L^2 norm is (recall that $\lambda = m + \delta$ where $|\delta| < \gamma$, and $\gamma < 1/10$)

$$\begin{aligned} \|G_\lambda\|^2 = \sum_{n \in \mathcal{N}_2} \frac{r_2(n)}{|n-\lambda|^2} = \frac{r_2(m)}{|m-\lambda|^2} + \sum_{\substack{n \in \mathcal{N}_2 \\ n \neq m}} \frac{r_2(n)}{(n-\lambda)^2} = \\ \frac{r_2(m)}{|m-\lambda|^2} + H'_m(m+\delta) + \sum_{n \notin I(\lambda)} \frac{r_2(n)}{|n-\lambda|^2} \leq \\ (2+O(\gamma)) \frac{r_2(m)}{|m-\lambda|^2} + O\left(\frac{\lambda^\epsilon}{\lambda^\eta}\right) = (2+O(\gamma)) \frac{r_2(m)}{|m-\lambda|^2} \end{aligned} \tag{36}$$

where the last inequality follows from Proposition 9, and that $r_2(n) \ll n^\varepsilon$. Recalling that f is positive, this implies that

$$\begin{aligned} \langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle = \frac{\sum_{n \in \mathcal{N}_2} \frac{W_f(n)}{(n-\lambda)^2}}{\sum_{n \in \mathcal{N}_2} \frac{r_2(n)}{(n-\lambda)^2}} \geq \frac{\frac{W_f(m)}{(m-\lambda)^2}}{(2+O(\gamma)) \frac{r_2(m)}{(m-\lambda)^2}} = \\ \frac{1}{2+O(\gamma)} \cdot \frac{W_f(m)}{r_2(m)} \to \frac{1}{2+O(\gamma)} \cdot \frac{1}{|\Xi(\infty)|} \sum_{\xi \in \Xi(\infty)} f(\xi). \end{aligned} \tag{37}$$

By choosing γ such that $2 + O(\gamma) > 0$, Theorem 1 is proved.

Remark 11. *We note that the above construction places mass at least* $1/2 + O(\gamma)$ *on the singular part.*

5.2. **Scarring in position space.** To simplify the notation, we use the following convention throughout this section: let $w := (0, 2) \in \mathbb{Z}^2$, and for λ a fixed "new" eigenvalue, and $v \in \mathbb{Z}^2$ define

$$c(v) := c_\lambda(v) = \frac{1}{|v|^2 - \lambda}, \qquad C(v, w) := c(v)c(v + w).$$

By the definition of G_λ and $\mathrm{Op}(a)$ we see that using the new notation (cf. (8), (11))

$$\mathrm{Op}(e_w)G_\lambda(x,x_0) = \sum_{v\in\mathbb{Z}^3} c(v)e^{iv\cdot x_0}e^{i(v+w)\cdot x},$$

and therefore
(38)

$$\langle \mathrm{Op}(e_w)g_\lambda, g_\lambda\rangle = \frac{e^{-iw\cdot x_0}\cdot \sum_{v\in\mathbb{Z}^2} c(v)c(v+w)}{\sum_{m\in\mathcal{N}_2}\frac{r_2(m)}{(m-\lambda)^2}} = \frac{e^{-iw\cdot x_0}\cdot \sum_{v\in\mathbb{Z}^2} C(v,w)}{\sum_{m\in\mathcal{N}_2}\frac{r_2(m)}{(m-\lambda)^2}}$$

As we aim to show that (38) is bounded from below in absolute value, we may assume that $x_0 = 0$. We will show that the sum in the numerator is essentially bounded from below by two terms in the sum, namely v such $|v| = |v+w|$.

In what follows, $\gamma \in (0, 1/10)$ is small (and to be determined later), $m = n^2 + 1$ will always denote an element of $\mathcal{M}_\gamma$ (recall that by construction, all elements of $\mathcal{M}_\gamma$ are of this form), and given n we define a vector $u \in \mathbb{Z}^2$ by

$$u := (n, -1).$$

For $\lambda \in \Lambda_\gamma$ let $m \in \mathcal{M}_\gamma$ be the corresponding nearby integer (i.e., $|m-\lambda| < \gamma$ by Proposition 9), set $R = \sqrt{\lambda}$, and let

$$C_m := \{v \in \mathbb{R}^2 : |v|^2 = m\}$$

denote the circle of radius $\sqrt{m}$ centered at the origin. Define

$$A_R = A_{R,w} := \{v \in \mathbb{R}^2 : |v| \in [R-|w|, R+|w|]\}$$

as the annulus of width $2|w|$ containing C_m, and let

$$A_R^* = A_{R,w}^* := \{v \in \mathbb{R}^2 : |v| \in [R-|w|, R+|w|], |v|^2 \neq m, |v+w|^2 \neq m\}.$$

The following Lemma will allow us to bound contribution of the negative terms in the sum in the numerator of the right hand side of (38).

Lemma 12. *If $C(v,w) < 0$ for $v \in \mathbb{Z}^2$, then $|v| \in [R-|w|, R+|w|]$. Furthermore, if we in addition have $|\langle v,w\rangle| \leq \frac{\sqrt{R}}{2}$, then $v = (\pm n, y)$ with $-3 \leq y \leq -1$ provided that R is sufficiently large.*

Proof. Since $C(v,w) < 0$ if and only if the line segment joining v and $v+w$ intersects C_R, the first assertion follows from the triangle inequality.

We now write $v = (x,y)$ for $x, y \in \mathbb{Z}$.

First case: If $|x| \geq n+1$, then

$$|v|^2 - \lambda \geq x^2 - \lambda \geq (n+1)^2 - \lambda = n^2 + 2n + 1 - \lambda \geq m + 1 - \lambda \geq 1 - \delta$$

and similarly $|v+w|^2 - \lambda \geq 1 - \delta$. Recalling that $|\delta| < \gamma \leq 1/10$ we find that $C(v,w) > 0$.

Second case: Assume that $|x| \leq n-1$. We note that $C(v,w) < 0$ implies that either $|v|^2 > \lambda$, or that $|v+w|^2 > \lambda$.

Now, if $|v|^2 > \lambda$, then

$$|v|^2 = x^2 + y^2 > \lambda = m + \delta = n^2 + 1 + \delta$$

so,

$$y^2 > n^2 + 1 + \delta - (n-1)^2 \geq n.$$

Consequently, $|y| \geq \sqrt{n} > \sqrt{R}/2$, hence $|\langle v, w\rangle| = 2|y| > \sqrt{R}$ and the claim is vacuous.

On the other hand, if $|v+w|^2 > \lambda$ then, as $x^2 \leq (n-1)^2$,

$$|v+w|^2 = x^2 + (y+2)^2 > \lambda = m + \delta = n^2 + 1 + \delta$$

so $|y| \geq \sqrt{n}$, and as before $|\langle v, w\rangle| > \sqrt{R}/2$; again the claim is vacuous.

Third case: For $|x| = n$, since $|\delta| = |\lambda - m| < 1/10$ we find that

$$|v|^2 - \lambda = n^2 + y^2 - \lambda = m - \lambda + y^2 - 1 = -\delta + y^2 - 1,$$

and

$$|v+w|^2 - \lambda = n^2 + (y+2)^2 - \lambda = -\delta + (y+2)^2 - 1$$

so both $c(v), c(v+w)$ are positive for $v = (\pm n, y)$ if $y \leq -4$ or $y \geq 2$. $\square$

In light of Lemma 12, we consider the following three sets of points $v \in \mathbb{Z}^2$:

$$\begin{aligned}
V_1 &:= \left\{ v \in \mathbb{Z}^2 : v = (\pm n, y), -3 \leq y \leq 1 \right\} \\
V_2 &:= \left\{ v \in \mathbb{Z}^2 : C(v,w) < 0, \substack{|v|^2, |v+w|^2 \neq m, \\ \sqrt{R}/2 \leq |\langle v,w\rangle| \leq 3R} \right\} \\
V_3 &:= \left\{ v \in \mathbb{Z}^2 : C(v,w) < 0, \substack{|v|^2 = m \text{ or } |v+w|^2 = m, \\ \sqrt{R}/2 \leq |\langle v,w\rangle| \leq 3R} \right\}.
\end{aligned}$$

Notice that these three sets, for R sufficiently large, cover all $v \in \mathbb{Z}^2$ such that $C(v,w) < 0$, because if $C(v,w) < 0$, then $|\langle v, w\rangle| \leq (R + |w|)|w| < 3R$ for $R > 4$. For ease of notation, we make the following definitions: For a finite set $X \subset \mathbb{R}$, define $\operatorname{argmin}_X(|x|)$ as the smallest $x \in X$ which minimizes $|x|$, and for $v \in \mathbb{Z}^2$ and w a "short vector" as before, let

$$\begin{aligned}
\mathrm{Nbr}_w(v) &:= \{C(v-w,w) + C(v,w), C(v,w) + C(v+w,w)\} \\
S_w(v) &:= \operatorname{argmin}_{\mathrm{Nbr}_w(v)}(|x|).
\end{aligned}$$

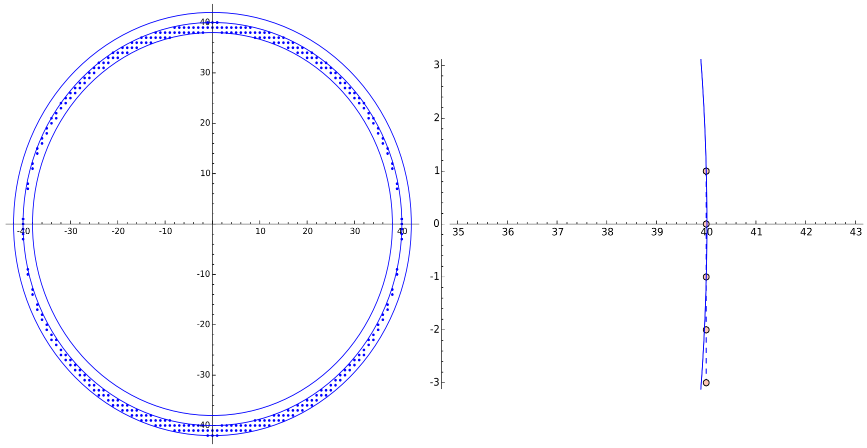

FIGURE 1. An illustration of setting of Lemma 12. For $m = 40^2 + 1$ and (say) $\lambda = 0.1$, **only** the lattice points with $C(v, w) < 0$ are plotted. On the right plot we zoomed around the point (40,0). Notice that the only points near $(40, \pm 1)$ lie on the line $x = 40$.

Corollary 13. *For any* $\delta \in (-1/10, 1/10)$*, as* $R \to \infty$

$$\sum_{v \in \mathbb{Z}^2} C(v, w) \geq \frac{2}{\delta^2} + O(\frac{1}{\delta}) + \sum_{v \in V_2} S_w(v) \tag{39}$$

Proof. We first notice that if R is large enough, only one sign change can occur for $|v + tw|^2 - \lambda$ when $t \in \mathbb{R}$ is bounded, so if $C(v, w) < 0$ then both $C(v - w, w), C(v + w, w) > 0$. Also, as mentioned above, by Lemma 12, if $C(v, w) < 0$ then v is in either V_1, V_2 or V_3 for R large. Therefore, after removing only positive terms, we find that

$$\sum_{v \in \mathbb{Z}^2} C(v, w) \geq \sum_{v \in V_1} C(v, w) + \sum_{v \in V_3} C(v, w) + \sum_{v \in V_2} S_w(v) = \tag{40}$$

$$2 \sum_{y=-3}^{1} C((n, y), w) + \sum_{v \in V_3} C(v, w) + \sum_{v \in V_2} S_w(v)$$

Now, by definition of $C(v, w)$, for the first sum we have that

$$\sum_{y=-3}^{1} C((\pm n, y), w) = \tag{41}$$

$$\frac{1}{(-\delta)^2} + 2\left(\frac{1}{(3-\delta)(-1-\delta)} - \frac{1}{(8-\delta)\delta}\right) = \frac{1}{\delta^2} + O\left(\frac{1}{\delta}\right).$$

(Note that when $\delta \to 0$, the dominant term $1/\delta^2$ comes from the term $y = -1$.)

For $v \in V_3$ and $|v|^2 = m$, we have $|\langle v, w\rangle| \gg \sqrt{R}$, and so (recall that $m - \lambda = -\delta$)

$$||v+w|^2 - \lambda| = ||v|^2 + 2\langle v, w\rangle + |w|^2 - \lambda| = |2\langle v, w\rangle + |w|^2 - \delta| \gg \sqrt{R}.$$

and $C(v,w) \ll \frac{1}{\delta\sqrt{R}}$. Using a similar argument we get that $C(v,w) \ll \frac{1}{\delta\sqrt{R}}$ if $|v+w|^2 = m$. Therefore, since $r_2(m)$ is bounded,

$$\sum_{v \in V_3} C(v,w) \ll \sum_{v \in V_3} \frac{1}{\delta\sqrt{R}} \ll \frac{1}{\delta\sqrt{R}}$$

and (39) follows. □

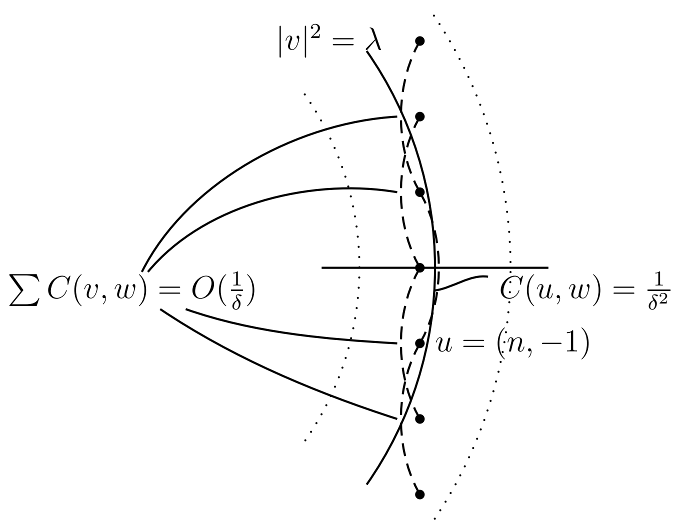

Figure 2. The main contribution in Corollary 13 is seen here. $C(u,w) = c(u)c(u+w) = \frac{1}{\delta^2}$, and all other points v with $C(v,w) < 0$ contribute $O(\frac{1}{\delta})$.

The contribution from the remaining, more subtle, terms are treated by “pairing off” negative summands with positive ones, and thereby getting some extra savings.

Lemma 14. *Let $v \in V_2$ be an element such that $|\langle v, w\rangle| \geq R^{1/3}$. Then $S_v(w) \ll \frac{\log^2 |\langle v,w\rangle|}{|\langle v,w\rangle|^2}$, as $R \to \infty$.*

Proof. For notational convenience, we put $B := B_v = |\langle v, w\rangle|$. We split the proof into two cases.

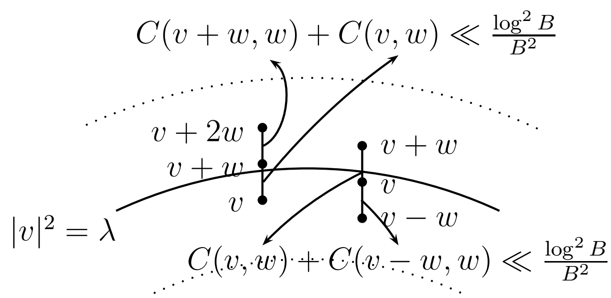

FIGURE 3. An illustration of Lemma 14. For $v \in V_2$, at least one of the terms $C(v,w)+C(v+w,w)$ or $C(v,w)+C(v-w,w)$is $\ll \frac{\log^2 B}{B^2}$

First case: Here we assume that $||v|^2-\lambda| \leq ||v+w|^2-\lambda|$. If $||v|^2-\lambda| \geq B/\log B$, then

$$|c(v)c(v+w)| \leq \frac{\log^2 B}{B^2}.$$

We may therefore assume that $||v|^2-\lambda| \leq B/\log B$. Now,

$$|v+w|^2-\lambda = |v|^2+2\langle v,w\rangle+|w|^2-\lambda = |v|^2-\lambda \pm 2B+|w|^2$$

and similarly $|v-w|^2-\lambda = |v|^2-\lambda-(\pm 2B)+|w|^2$, hence

$$C(v,w)+C(v-w,w) = c(v)\cdot c(v+w)+c(v-w)\cdot c(v) = \tag{42}$$
$$= \frac{1}{|v|^2-\lambda}\left(\frac{2(|v|^2+|w|^2-\lambda)}{(|v|^2+2B+|w|^2-\lambda)(|v|^2-2B+|w|^2-\lambda)}\right).$$

(note that the two $\pm 2B$ terms above occur with *opposite* signs). Recalling the assumption $|v|^2 \neq m$, together with $|w|^2=4$, we find that $||v|^2-\lambda| \geq 1/2$ (note that $|\lambda-m| \leq \delta \leq 1/10$ by our assumption on δ), and this together with (42) shows that

$$C(v,w)+C(v-w,w) \ll \frac{1}{(|v|^2+2B+|w|^2-\lambda)(|v|^2-2B+|w|^2-\lambda)}.$$

Since we assume that $||v|^2-\lambda| \leq B/\log B$, we find that

$$||v|^2 \pm 2B+|w|^2-\lambda| \gg B$$

and thus $C(v,w)+C(v-w,w) \ll 1/B^2$.

Second case. Here we assume that $||v|^2-\lambda| > ||v+w|^2-\lambda|$. This case follows by a similar argument, except for showing that

$$|c(v+w)(c(v)+c(v+2w))| \ll \frac{\log^2 B}{B^2}.$$

□

Corollary 15. *As $R \to \infty$, we have*

$$\sum_{v \in V_2} S_w(v) = \sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [R^{1/3}, 3R]}} S_w(v) = o(1).$$

Proof. Write

$$\sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [R^{1/3}, 3R]}} S_w(v) = \sum_{\substack{k \in \mathbb{N} \\ R^{1/3} \le 2^k < R/\log R}} \sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [2^k, 2^{k+1})}} S_w(v) + \sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [R/\log R, 3R]}} S_w(v) \tag{43}$$

Since the number of lattice points in A_R satisfying $\langle v, w\rangle \in I$ is $O(|I||w|)$ for any interval $I \subset [-R/\log R, R/\log R]$, we get by Lemma 14 that

$$\sum_{\substack{k \in \mathbb{N} \\ R^{1/3} \le 2^k < R/\log R}} \sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [2^k, 2^{k+1})}} S_w(v) \ll \sum_{2^k \ge R^{1/3}} 2^k \left(\frac{k}{2^k}\right)^2 \ll \frac{1}{R^{1/3-\varepsilon}} = o(1)$$

and

$$\sum_{\substack{v \in A_R^* \\ C(v,w)<0 \\ |\langle v,w\rangle| \in [R/\log R, 3R]}} S_w(v) \ll R \frac{\log^2 R}{(R/\log R)^2} = \frac{\log^4 R}{R} = o(1)$$

□

5.2.1. *Conclusion.* We can now conclude the proof of Theorem 1 by proving that for $f(x) = e^{\pi i \langle x, w\rangle} \in C^\infty(\mathbb{T}^2)$,

$$\lim_{\lambda \in \Lambda'} \langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle > 0 \tag{44}$$

By (38) (recall that we may assume that $x_0 = 0$),

$$\langle Op(e_w) g_\lambda, g_\lambda \rangle = \frac{\sum_{v \in \mathbb{Z}^2} C(v,w)}{\sum_{m \in \mathcal{N}_2} \frac{r_2(m)}{(m-\lambda)^2}}$$

By corollaries 13 and 15

$$\sum_{v\in\mathbb{Z}^2} C(v,w) \geq \frac{2}{\delta^2} + O(\frac{1}{\delta}) \tag{45}$$

On the other hand, by Proposition 9

$$\sum_{n\in\mathcal{N}_2} \frac{r_2(n)}{(n-\lambda)^2} = \frac{r_2(m)}{(m-\lambda)^2} + H'_m(\lambda) \leq (2+O(\gamma))\frac{r_2(m)}{(m-\lambda)^2}$$

and, recalling that $r_2(m)$ is bounded, we can choose γ small enough (recall that $|m-\lambda| = |\delta| \leq \gamma$) so that

$$\langle \mathrm{Op}(e_w)g_\lambda, g_\lambda\rangle \geq \frac{\frac{2}{\delta^2} + O(\frac{1}{\delta})}{(2+O(\gamma))\frac{r_2(m)}{\delta^2}} = \frac{2+O(\delta)}{(2+O(\gamma))r_2(m)} = \frac{2+O(\gamma)}{(2+O(\gamma))r_2(m)}$$

is *uniformly* bounded from below.

6. Proof of Theorem 3

Recall first the setting proved in Proposition 8: There exist an integer $H \geq 1$ with the property that for all sufficiently large R there exist an integer $h \in (0, H^2)$ such that

$$|\{n \in \mathcal{N}_2 : n \leq x, \quad 0 < r_2(n) \ll R^H, \quad r_2(n+h) \geq R\cdot r_2(n)\}| \gg_R x/(\log x)^H$$

as $x \to \infty$.

As noted in Remark 10, if $r_2(m+h) \geq R\cdot r_2(m) > 0$ for some integer h such that $0 < h < H$, then there exists a new eigenvalue $\lambda = m + \delta_0$ with

$$\delta_0 \ll 2H/\sqrt{R}, \qquad H'_m(\lambda) \ll \frac{r_2(m)}{(m-\lambda)^2}.$$

The argument in Section 5.1 then shows that λ gives rise to a momentum scar provided $r_2(m)$ is also bounded. Proposition 8 then gives, upon choosing R sufficently large, that the number of such $m \leq x$ is $\gg x/(\log x)^H$.

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